JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 44, 447–463 (1973)

Representations of Solutions of Abstract Second Order Differential Equations with Applications to Stiff Equations and Singular Perturbations

J. EISENFELD

Department of Mathematics, University of Texas, Arlington, Texas 76010

Submitted by Peter D. Lax

Received October 31, 1972

The purpose of this paper is to demonstrate the applicability of the theory of operator equations to the theory of abstract differential equations. Attention is focused on the representation and approximation of solutions. Particular topics considered are existence–uniqueness, stiff equations, and singular perturbations.

1. INTRODUCTION

Consider the differential equation

$$A \frac{d^2}{dt^2} x(t) + B \frac{d}{dt} x(t) + Cx(t) = 0, \qquad -\infty < t < \infty, \tag{1.1}$$

where A, B and C are t-independent linear operators defined in and mapping into a (real or complex) Banach space. Suppose that each of the linear operators Z_1, Z_2 is a root of the operator equation

$$AZ^2 + BZ + C = 0, \tag{1.2}$$

then it is easy to see, at least formally, that for any pair of constant vectors, $c_1, c_2 \in \mathscr{B}$ the vector valued function,

$$x(t) = e^{tZ_1}c_1 + e^{tZ_2}c_2, \tag{1.3}$$

is a solution of (1.1). We will show that, if the pair Z_1, Z_2 *Q-complete*, i.e., $Z_2 - Z_1$ has a left inverse Q in $\mathscr{R}$, the Banach algebra of bounded linear operators on $\mathscr{B}$ into itself, then the constant vectors c_1, c_2 may be chosen as to satisfy a certain class of initial conditions. Furthermore, with the aid of a

Q-complete pair of roots, one can deal with initial value conditions, end-point conditions, and inhomogeneous terms, in a manner completely similar to that used in the simple scalar case. Therefore, the solving of abstract second order equations reduces to the finding of a pair of roots of the corresponding characteristic equation (1.2). The latter has been accomplished under various sets of assumptions cf. [4, 5, 8, 9].

The formula (1.3) and other similar statements which are given below have several applications. First of all, it demonstrates the existence of a solution. We will, give an example for which one has existence of solutions to the abstract Cauchy problem with certain initial data but not uniqueness.

Second of all, suppose that it is possible to construct from the coefficient operators an approximate pair of roots $W_i \sim Z_i$, $i = 1, 2$, as is the case (cf. Lemma 3 below) when

$$4 \| B^{-1}A \| \| B^{-1}C \| < 1. \tag{1.4}$$

Then the replacing of Z_1, Z_2 by W_1, W_2 provides an approximate solution. When the left side of (1.4) is very small the spectrum of the characteristic equation,

$$(\lambda^2 A + \lambda B + C)\, x = 0, \tag{1.5}$$

separates into two sets Λ_1 and Λ_2 where Λ_1 (the spectrum of Z_1) lies close to the origin and Λ_2 (the spectrum of Z_2) lies far from the origin [4]. Equations exhibiting such characteristics are sometimes called stiff equations and are difficult to solve, numerically, even for finite systems of low order differential equations [3, 10]. It is anticipated that methods developed here will be applied advantageously to the theory of stiff equations.

Another source of applications is to the theory of singular perturbations. Consider, for example, the case

$$A = \epsilon A_1, \tag{1.6}$$

where ϵ is a small positive parameter. The condition (1.4) is satisfied for $\epsilon < 1/(4 \| B^{-1}A_1 \| \| B^{-1}C \|)$, and we will then construct approximate roots $Z_i^{(n)}(\epsilon)$ such that

$$\| Z_i(\epsilon) - Z_i^{(n)}(\epsilon) \| = O(\epsilon^{n+1}), \qquad \epsilon \to 0, \quad i = 1, 2. \tag{1.7}$$

This will furnish asymptotic approximations of the solution.

In this paper we consider only cases for which the operators in question are bounded. However, it is possible, using the theory of semigroups (cf. [7]), to extend our results to the case of unbounded operators.

2. Representations of Solutions in Terms of the Operator Roots of the Characteristic Equation

Let Z_1 and Z_2 be roots in $\mathscr{R}$ of (1.2). Then certainly the vector valued function in (1.3) is a solution of (1.1) (cf. [7]). Suppose x_0, x_0' are given vectors in $\mathscr{B}$. We want then to determine c_1 and c_2 in (1.3) so that

$$x(0) = x_0, \qquad \frac{d}{dt}(x_0) = x_0'. \tag{2.1}$$

Suppose that there is an operator Q in $\mathscr{R}$ such that

$$Q(Z_2 - Z_1) = I, \tag{2.2}$$

where I is the identity operator on $\mathscr{B}$. Then

$$c_1 = x_0 - c_2, \qquad c_2 = Q(x_0' - Z_1 x_0), \tag{2.3}$$

provided

$$x_0' - Z_1 x_0 \in \operatorname{Range}(Z_2 - Z_1). \tag{2.4}$$

Next, consider the inhomogeneous equation

$$A \frac{d^2}{dt^2} x(t) + B \frac{d}{dt} x(t) + Cx(t) = y(t), \qquad t > 0, \tag{2.5}$$

where the vector valued function $y(t)$ is defined and continuous for $t \geqslant 0$. Let us assume further that

$$A(Z_2 - Z_1) = H, \tag{2.6}$$

where H has an inverse (both left and right) $H^{-1} \in \mathscr{R}$ (such a condition is satisfied under the hypothesis of Lemma 3). Then

$$Q = H^{-1}A, \tag{2.7}$$

and one can verify that the variation of parameters formula,

$$x_p(t) = \int_0^t [e^{(t-\tau)Z_1} - e^{(t-\tau)Z_2}]\, H^{-1} y(\tau)\, d\tau, \tag{2.8}$$

provides for a particular solution $x_p(t)$ of (2.5). A solution of (2.5) which satisfies the initial conditions of (2.1) is given by

$$x(t) = e^{tZ_1}c_1 + e^{tZ_2}c_2 + x_p(t), \tag{2.9}$$

where c_1 and c_2 are found from (2.3).

Now consider the two point boundary conditions

$$F\frac{d}{dt}x(0)+G_0x(0)=0, \qquad F_1\frac{d}{dt}x(1)+G_1x(1)=0, \tag{2.10}$$

where F_i, $G_i \in \mathscr{R}$, $i=0,1$. Then the boundary value problem (1.1)–(2.10) has a solution of the form (1.3) if and only if the matrix operator

$$\mathscr{G}=\begin{pmatrix}(F_0Z_1+G_0) & (F_1Z_2+G_1)\\ (F_0Z_1e^{Z_1}+G_0e^{Z_1}) & (F_1Z_2e^{Z_2}+G_1e^{Z_2})\end{pmatrix} \tag{2.11}$$

defined on the product space $\mathscr{B}\oplus\mathscr{B}$ has a nontrivial null space $\mathscr{N}$. If $c_1\oplus c_2\in\mathscr{N}$ then a solution is given by (1.3).

Suppose the coefficients A, B, C and the boundary operators F_i, G_i, $i=0,1$ depend on a parameter λ. Then one has a nontrivial solution only at an eigenvalue of the expression $\mathscr{G}(\lambda)$. If λ_0 is not in the spectrum of $\mathscr{G}$, i.e., when $\mathscr{G}(\lambda_0)$ is one to one and onto then, by means of the variation of parameters formula (2.8), one determines a solution of

$$A\frac{d^2}{dt^2}x(t)+B\frac{d}{dt}x(t)+Cx(t)=y(t), \qquad 0\leqslant t<1, \tag{2.12}$$

which satisfies the boundary conditions (2.10). Here $y(t)$ is assumed to be defined and continuous on $[0, 1]$.

Consider, for example, the boundary conditions

$$x(0)=x(1)=0. \tag{2.13}$$

We are still thinking of the coefficient operators depending on the parameter λ. The condition for finding a nontrivial solution of the form (1.3) of the homogeneous problem (1.1)–(2.13) is that $\lambda=\lambda_0$ be an eigenvalue of the expression

$$M(\lambda)=e^{Z_2(\lambda)}-e^{Z_1(\lambda)}. \tag{2.14}$$

Then the substitution $c_1=-c_2=v$, where v is an eigenvector belonging to λ_0, provides a solution of the form (1.3). If, on the other hand, λ_0 is not in the spectrum of M then a solution of (2.13)–(2.12) is given by

$$\begin{aligned} x(t)=[e^{tZ_1(\lambda_0)}&-e^{tZ_2(\lambda_0)}]\,M^{-1}(\lambda_0)\\ &\times\int_0^1[e^{(1-\tau)Z_1(\lambda_0)}-e^{(1-\tau)Z_2(\lambda_0)}]\,H^{-1}(\lambda_0)\,y(\tau)\,d\tau \\ &+\int_0^t[e^{(t-\tau)Z_1(\lambda_0)}-e^{(t-\tau)Z_2(\lambda_0)}]\,H^{-1}(\lambda_0)\,y(\tau)\,d\tau. \end{aligned} \tag{2.15}$$

3. Contour Integral Representations

In this section let us assume further that A has an inverse in $\mathscr{R}$. Then $Z_1 - Z_2$ has an inverse in $\mathscr{R}$. Then one has the following formula for functions h, analytic on a Cauchy domain Γ (with boundary $\partial\Gamma$) which contains the spectra of Z_1 and Z_2 (cf. [4]).

$$h(Z_2) - h(Z_1) = \left(\frac{1}{2\pi i}\int_{\partial\Gamma} h(\lambda)\,(\lambda^2 A + \lambda B + C)^{-1}\,d\lambda\right) H. \tag{3.1}$$

In particular, taking $h(\lambda) = e^{(t-\tau)\lambda}$ we may express the particular solution (2.8),

$$x_p(t) = -\frac{1}{2\pi i}\int_0^t\int_{\partial\Gamma} e^{(t-\tau)}(\lambda^2 A - \lambda B + C)^{-1}\,y(\tau)\,d\lambda\,d\tau, \tag{3.2}$$

the expression (2.14),

$$M(\mu) = \frac{1}{2\pi i}\int_{\partial\Gamma(\mu)} e^{\lambda}(\lambda^2 A + \lambda B + C)^{-1}\,d\lambda, \tag{3.3}$$

and there is a similar representation of the solution (2.15). Moreover, using the Taylor–Dunford integral representation of functions of operators, we express

$$e^{tZ_i} = \frac{1}{2\pi i}\int_{\partial\Gamma} e^{\lambda t}(\lambda I - Z_i)^{-1}\,d\lambda, \qquad i = 1, 2. \tag{3.4}$$

We will also make use of the following result [4].

Lemma 1. *Let X be an operator in $\mathscr{R}$, and let Γ be a bounded Cauchy domain containing the spectrum of X in its interior. Let $h(\lambda)$ be analytic in Γ and on its boundary $\partial\Gamma$ and let ν be the maximum modulus of $h(\lambda)$ on $\partial\Gamma$. Denote by γ the maximum of the norm of $(\lambda I - X)^{-1}$ on $\partial\Gamma$. Suppose $Y \in \mathscr{R}$ is such that*

$$\delta = \gamma\,\|X - Y\| < 1, \qquad |\partial\Gamma|\,\gamma\delta < 2\pi(1 - \delta). \tag{3.5}$$

Then Γ contains the spectrum of Y in its interior and

$$\|h(X) - h(Y)\| \leqslant \frac{|\partial\Gamma|}{2\pi}\,\frac{\nu\gamma\delta}{1 - \delta}. \tag{3.6}$$

It is not difficult to extend Lemma 2 so that it yields the following.

Lemma 2. *Let the hypothesis of Lemma 1 be satisfied except now let*

$$Y(\lambda) = X_1(\lambda) \tag{3.7}$$

be λ-dependent where $X_1(\lambda)$ is sufficiently small on $\partial\Gamma$ such that

$$\delta_1 = \max_{\lambda \in \partial\Gamma} \| X_1(\lambda) \| < 1. \tag{3.8}$$

Then $(\lambda I - Y(\lambda))^{-1}$ is defined on $\partial\Gamma$ and

$$\left\| \int_{\partial\Gamma} h(\lambda) \left[(\lambda I - Y(\lambda))^{-1} - (\lambda I - X)^{-1} \right] d\lambda \right\| \leqslant \frac{|\partial\Gamma| \nu\gamma\delta}{1 - \delta}. \tag{3.9}$$

4. Existence and Uniqueness: An Example

Let A be the left shift operator in l_1, i.e.,

$$A(\alpha_1, \alpha_2, ..., \alpha_n, ...) = (\alpha_2, \alpha_3, ..., \alpha_{n-1}, ...). \tag{4.1}$$

The equation,

$$A \frac{d^2}{dt^2} x(t) - \frac{d}{dt} x(t) = 0, \tag{4.2}$$

has a solution,

$$x(t) = \left(\frac{t^2}{2!}, \frac{t^2}{3!}, ..., \frac{t^{n+1}}{(n+1)!}, ... \right), \tag{4.3}$$

which satisfies the initial conditions

$$x(0) = \frac{d}{dt} x(0) = 0. \tag{4.4}$$

On the other hand, A has a right inverse A^r,

$$A^r(\alpha_1, \alpha_2, ..., \alpha_n, ...) = (0, \alpha_1, \alpha_2, ..., \alpha_{n+1}, ...), \tag{4.5}$$

and the characteristic equation,

$$AZ^2 - Z = 0, \tag{4.6}$$

has a pair of A-complete roots,

$$Z_1 = 0, \qquad Z_2 = A^r. \tag{4.7}$$

Therefore, solutions to the abstract Cauchy problem,

$$A \frac{d^2}{dt^2} x(t) - \frac{d}{dt} x(t) = y(t), \qquad x(0) = x_0, \qquad \frac{d}{dt} x(0) = x_0', \tag{4.8}$$

exist for

$$x_0' \in \operatorname{Range}(A^r) \tag{4.9}$$

but are not unique.

5. Construction of Roots

The following consists of various results proved in Sections 3 and 5 of [4].

LEMMA 3. *Let $\hat{A}_0\hat{C} \in \mathscr{R}$ such that*

$$4\,\|\hat{A}\|\,\|\hat{C}\| < 1, \tag{5.1}$$

and A has a right inverse $A^r \in \mathscr{R}$. Then

(1) *The operator equation,*

$$\hat{A}Z^2 + Z + \hat{C}, \tag{5.2}$$

has a pair of roots Z_1, $Z_2 \in \mathscr{R}$ which are Q-complete (cf. (5.17)),

(2) *Let*

$$\begin{aligned} d &= (1 - 4\,\|\hat{A}\|\,\|\hat{C}\|)^{1/2}, \qquad & a &= (1-d)/(2\,\|\hat{A}\|), \\ b &= (1+d)/(2\,\|\hat{A}\|), & \rho &= \|\hat{A}^r\|\,(1 + \|\hat{A}\|\,a), \end{aligned} \tag{5.3}$$

then

$$\|Z_1\| \leqslant a, \qquad \|Z_2\| \leqslant \min\{\rho, \|\hat{C}\|\,b\}; \tag{5.4}$$

(3) *Z_1 is the only root of* (5.2) *of norm $\leqslant b$;*

(4) *Z_2 has an inverse in $\mathscr{R}$ and*

$$\|Z_2^{-1}\| \leqslant b^{-1}; \tag{5.5}$$

(5) *Z_1 is compact (has an inverse in $\mathscr{R}$) if and only if $\hat{C}$ is compact (has an inverse in $\mathscr{R}$);*

(6) *the iterative sequence of operators,*

$$Z_1^{(0)} = -\hat{C}, \qquad Z_1^{(n+1)} = -\hat{A}(Z^{(n)})^2 - \hat{C}, \tag{5.6}$$

all have norm $\leqslant a$, converge to Z_1 and

$$\|Z_1^{(n)} - Z_1^{(m)}\| \leqslant (1-d)^n\,\|\hat{A}\|\,\|\hat{C}\|/d, \qquad m \geqslant n, \quad n = 0, 1,\ldots. \tag{5.7}$$

$$\|Z_1^{(n)} - Z_1\| \leqslant (1-d)^n\,\|\hat{A}\|\,\|\hat{C}\|/d, \qquad n = 0, 1,\ldots; \tag{5.8}$$

(7) *the iterative sequence of operators,*

$$X_1^{(0)} = -\hat{C}, \qquad X_1^{(n+1)} = -(X_1^{(n)})^2 \hat{A} - \hat{C}, \tag{5.9}$$

all have norm $\leqslant a$, *converge to an operator* $X_1 \in \mathscr{R}$ *and*

$$\| X_1^{(n)} - X_1^{(m)} \| \leqslant (1-d)^n \| \hat{A} \| \| \hat{C} \| / d, \qquad m \geqslant n, \qquad n = 0, 1, \ldots, \tag{5.10}$$

$$\| X_1^{(n)} - X_1 \| \leqslant (1-d)^n \| \hat{A} \| \| \hat{C} \| / d, \qquad n = 0, 1, \ldots, \tag{5.11}$$

(8) X_1 *is compact* (*has an inverse in* $\mathscr{R}$) *if and only if* $\hat{C}$ *is compact* (*has an inverse in* $\mathscr{R}$);

(9) *the root* Z_2 *is related to* X_1 *by*

$$Z_2 = -\hat{A}^r(I + X_1\hat{A}); \tag{5.12}$$

(10) *Let*

$$D = -X_1\hat{A} - \hat{A}Z_1, \tag{5.13}$$

then

$$\| D \| \leqslant 1 - d < 1, \tag{5.14}$$

thereby the operator,

$$H = -I + D, \tag{5.15}$$

has an inverse in $\mathscr{R}$ *and*

$$H^{-1} = -\sum_{n=0}^{\infty} D^n; \tag{5.16}$$

(11) $\hat{A}$, Z_1, Z_2 *and* Q *are related by*

$$\hat{A}(Z_2 - Z_1) = H, \qquad Q = H^{-1}\hat{A}. \tag{5.17}$$

6. Stiff Equations

Consider a system of second order equations of the form (1.1) where A and B have inverses in $\mathscr{R}$ (in the most common examples [3, 10] A, B, and C are matrices). We set

$$\hat{A} = B^{-1}A, \qquad \hat{C} = B^{-1}A \tag{6.1}$$

and measure the stiffness by

$$s = \| \hat{A} \| \| \hat{C} \| . \tag{6.2}$$

When $s < \frac{1}{4}$ the characteristic equation (1.2) has a pair of complete

(I-complete) roots; Z_1 and Z_2, and approximate roots which are constructed from A, B, and C: $Z_1^{(n)}$,

$$Z_2^{(n)} = -\hat{A}^{-1}(I + X_1^{(n)}\hat{A}), \tag{6.3}$$

all furnished by Lemma 3. Naturally, we approximate the solutions to initial value problems and end-point boundary value problems with inhomogeneous terms by replacing Z_1, Z_2 by their corresponding approximations in the above representations. This leads to the problem of approximating

$$e^{tZ_i}, e^{tZ_2}, (Z_2 - Z_1)^{-1}. \tag{6.4}$$

Since $\| Z_1 \| \leqslant a$, $\| Z_1^{(n)} \| \leqslant a$ we have by expanding e^{tZ_1} and e^{tZ_2} in power series and taking account of (5.8) and (5.11):

$$\| e^{tZ_1} - e^{tZ_1^{(n)}} \| \leqslant ae^{ta}(1 - d)^n \, s/d = e^{ta}O(s^{n+1}). \tag{6.5}$$

Similarly,

$$\| e^{tZ_1} - e^{tZ_2^{(n)}} \| \leqslant \rho e^{te}(1 - d)^n \| \hat{A} \| \| \hat{A}^{-1} \| \, s/d = e^{t\rho}O(s^{n+1}). \tag{6.6}$$

Let

$$D^{(n)} = -X_1^{(n)}\hat{A} - \hat{A}Z_1^{(n)}. \tag{6.7}$$

Then from (5.13)

$$\| D^{(n)} - D \| \leqslant (1 - d)^n \| \hat{A} \|^2 \| C \| / d. \tag{6.8}$$

Since $\| Z_1^{(n)} \| \| X_1^{(n)} \| \leqslant a$, $\| D_n \| \leqslant 1 - d < 1$ and we may define

$$(H^{(n)})^{-1} = -\sum_{j=0}^{\infty} (D^{(n)})^j. \tag{6.9}$$

Then from (5.16),

$$\| H^{-1} - (H^{(n)})^{-1} \| \leqslant (1 - d)^n \| \hat{A} \|^2 \| \hat{C} \| / d^3. \tag{6.10}$$

Let us consider, as in [3], equations of the type

$$A \frac{d^2}{dt^2} x(t) + \alpha \frac{d}{dt} x(t) + Cx(t) = y(t), \qquad 0 < t < 1, \tag{6.11}$$

where α is a large constant. Here

$$s = \| A \| \| C \| / \alpha^2 \to 0, \qquad \text{as} \quad \alpha \to \infty, \tag{6.12}$$

$$d = 1 - 2s + O(s^2) \to 1 \qquad \text{as} \quad \alpha \to \infty, \tag{6.13}$$

$$1 - d = 2s + O(s^2) \to 0 \qquad \text{as} \quad \alpha \to \infty, \tag{6.14}$$

$$a = \| C \| / \alpha + O(1/\alpha^2) \to 0 \qquad \text{as} \quad \alpha \to \infty, \tag{6.15}$$

$$\rho = \| A^{-1} \| (1 - \| A \| a) \to \| A^{-1} \| \qquad \text{as} \quad \alpha \to \infty, \tag{6.16}$$

and the right side of inequality (6.70),

$$p = (1-d)^m \| \hat{A} \|^2 \| \hat{C} \| / d^3 = O((1/\alpha)^{2n+5}) \to 0 \qquad \text{as} \qquad \alpha \to \infty. \tag{6.17}$$

One sees that the estimate (6.5), (6.6), and (6.10) provide for reasonably good approximations of solutions of equations of the type (6.11). They will also provide good approximations for equations of the type

$$A \frac{d^2}{dt} x(t) + B \frac{d}{dt} x(t) + \epsilon C x(t) = y(t), \qquad 0 < t < 1, \tag{6.18}$$

where ϵ is small. The case when A is small while B and C are not is another matter. There is also the problem of obtaining uniform approximations in t when t varies over an infinite interval. These matters will be considered below in the context of singular perturbations.

7. Singular Perturbations

Consider the Eq. (1.1) where the leading coefficient depends on a small positive parameter as in (1.6). We assume here, as in the previous section, that $A = A(\epsilon)$ and B have inverses in $\mathscr{R}$ and we define $\hat{A} = \hat{A}(\epsilon)$ and $\hat{C}$ by (6.1). Also, let

$$\hat{A}_1 = B^{-1} A_1 . \tag{7.1}$$

For $\epsilon < \frac{1}{4} \| \hat{A}_1 \| \| \hat{C} \|$ we obtain from Lemma 3 the roots and approximate roots for (1.2)–(1.6) which are now ϵ dependent. We have for example from (5.6)

$$\begin{gathered} Z_1^{(0)} = -\hat{C}, \qquad Z_1^{(1)} = -\hat{C} - \epsilon \hat{A}_1 \hat{C}^2, \\ Z_1^{(2)} = -\hat{C} - \epsilon \hat{A}_1 \hat{C}^2 - \epsilon^2 A_1 (\hat{A}_1 \hat{C}^3 + \hat{C} \hat{A}_1 \hat{C}^2) - \epsilon^3 \hat{A}_1 (\hat{A}_1 \hat{C}^2)^2. \end{gathered} \tag{7.2}$$

It is not difficult to see that the $Z_1^{(n)}$'s are polynomials in ϵ of degree $\geqslant n$. Let $W_1^{(n)}$ be the polynomial in ϵ of degree n such that

$$\| W_1^{(n)} - Z_1^{(n)} \| = O(\epsilon^{n+1}), \qquad \epsilon \to 0. \tag{7.3}$$

Since from (5.3)

$$d = 1 - 2 \| \hat{A}_1 \| \| \hat{C} \| \epsilon + O(\epsilon^2) \to 1 \qquad \text{as} \qquad \epsilon \to 0, \tag{7.4}$$

the estimate (5.8) shows that $Z_1^{(n)}$ converge to Z_1 uniformly in ϵ and

$$\| Z_1 - Z_1^{(n)} \| = O(\epsilon^{n+1}), \qquad \epsilon \to 0. \tag{7.5}$$

Thus, Z_1 is analytic in ϵ [7], i.e.,

$$Z_1 = \sum_{i=0}^{\infty} J_i \epsilon^i, \tag{7.6}$$

and

$$W_1^n = \sum_{i=0}^{n} J_i \epsilon^i, \qquad n = 0, 1, \ldots. \tag{7.7}$$

Similarly, $X_1^{(n)}$ is a polynomial in ϵ of degree $\geqslant n$ and if we let $Y_1^{(n)}$ be the polynomial of degree n such that

$$\| Y_1^{(n)} - X_1 \| = O(\epsilon^{n+1}), \qquad \epsilon \to 0, \tag{7.8}$$

then

$$X_1 = \sum_{i=0}^{\infty} K_i \epsilon^i \tag{7.9}$$

and

$$Y_1^{(n)} = \sum_{i=0}^{n} K_i \epsilon^i. \tag{7.10}$$

Thus, the root Z_2 has the Laurant expansion

$$Z_2 = -\hat{A}_1^{-1} \epsilon^{-1} - \sum_{i=0}^{\infty} \hat{A}_1^{-1} K_i \hat{A}_i \epsilon^i. \tag{7.11}$$

In the jargon of singular perturbations [2], we have the outer approximation and the inner approximation. The outer approximation is obtained by setting $\epsilon = 0$ in (1.1)–(1.6) thereby obtaining the outer approximate equation

$$B \frac{d}{dt} x_1(t) + C x_1(t) = 0. \tag{7.12}$$

To find the inner approximate equation: we set

$$\tau = t/\epsilon; \tag{7.13}$$

let $\epsilon = 0$ in the resulting equation

$$A_1 \frac{d^2}{d\tau^2} \tilde{x}(\tau) + B \frac{d}{d\tau} \tilde{x}(\tau) + \epsilon C \frac{d}{d\tau} \tilde{x}(\tau) = 0; \tag{7.14}$$

and then substitute $t = \tau\epsilon$ to obtain

$$\epsilon A_1 \frac{d^2}{dt^2} x_2(t) + B \frac{d}{dt} x_2(t) = 0. \tag{7.15}$$

Solutions to (7.11) are given by

$$x_1(t) = e^{-t\hat{C}}c_1\,, \tag{7.16}$$

where c_1 is an arbitrary constant vector while a solution to (7.15) is given by

$$x_2(t) = e^{-t\hat{A}_1/\epsilon}c_2\,, \tag{7.17}$$

where c_2 is an arbitrary constant vector.

On the other hand, from Section 2, the general solution of (1.1)–(1.6) is

$$x(t) = e^{-tZ_1}c_1 + e^{-tZ_2}c_2\,. \tag{7.18}$$

Since

$$Z_1 \sim -\hat{C}, \qquad Z_2 \sim -\hat{A}_1/\epsilon \qquad \text{as} \qquad \epsilon \to 0, \tag{2.19}$$

the root Z_1 corresponds to the outer approximations while Z_2 corresponds to the inner approximation.

As one might expect

$$a = \| \hat{C}_1 \| + O(\epsilon) \to \| \hat{C}_1 \| \qquad \text{as} \qquad \epsilon \to 0, \tag{7.20}$$

$$\rho = 1/\| \hat{A}_1 \| \,\epsilon + O(\epsilon) \to \infty \qquad \text{as} \qquad \epsilon \to 0; \tag{7.21}$$

therefore, the estimate (6.6) (replacing $\epsilon = O(s)$ by (s) is not advantageous in this case. However, we may resort to the contour integral representations of Section 3.

Let Γ_2 be a Cauchy domain containing the spectrum of $\hat{A}_1$ in its interior, and let

$$\hat{\Gamma}_2 = \{\hat{\lambda} = -\lambda/\epsilon \mid \lambda \in \Gamma_2\}. \tag{7.22}$$

Then $\hat{\Gamma}_2$ contains the spectrum of $-\hat{A}_1/\epsilon$ in its interior. Suppose

$$\gamma_2 = \max_{\lambda \in \partial\Gamma_2} \|(\lambda I - \hat{A}_1)^{-1}\|\,. \tag{7.23}$$

Then

$$\hat{\gamma}_2 = \max_{\hat{\lambda} \in \partial\hat{\Gamma}_2} \|(\hat{\lambda} I + \hat{A}_1/\epsilon)^{-1}\| = \epsilon\gamma_2\,. \tag{2.24}$$

Let $Z_2^{(n)}$ be defined as in (6.3), and let

$$\begin{aligned} \delta_2^{(n)} = \| Z_2 - Z_2^{(n)} \| \hat{\gamma}_2 &\leqslant (1-d)^n \epsilon^2 \| \hat{A}_1^{-1} \| \\ &\| \hat{A}_1 \|^2 \| \hat{C} \| \gamma_2/d = O(\epsilon^{n+2}) \end{aligned} \tag{7.25}$$

(cf. (5.11) and (7.4)). For fixed $n \geqslant 0$ let $\epsilon_n > 0$ be chosen sufficiently small so that

$$\delta_2^{(n)} < 1, \qquad | \partial\hat{\Gamma}_2 | \hat{\gamma}_2 \delta_2^{(n)} < 2\pi(1 - \delta_2^{(n)}), \qquad \epsilon < \epsilon_n; \tag{7.26}$$

note that

$$| \partial\hat{\Gamma}_2 | \hat{\gamma}_2 \delta_2^{(n)} = O(\epsilon^{n+2}), \qquad \epsilon \to 0. \tag{7.27}$$

Next, let for fixed $n \geqslant 0$

$$Z_2^{(-1)} = -\hat{A}_1^{-1}/\epsilon, \qquad \delta^{(-1)} = \| Z_2^{(n)} - Z_2^{(-1)} \| \hat{\gamma}_2(\epsilon) = O(\epsilon), \qquad \epsilon \to 0, \tag{7.28}$$

and chose $\omega_2^{(n)} \leqslant \epsilon_n$ such that

$$\delta^{(-1)} < 1, \qquad | \partial\hat{\Gamma}_2 | \hat{\gamma}_2 \delta^{(-1)} < 2\pi(1 - \delta^{(-1)}), \qquad \text{for } \epsilon < \omega_2^{(n)}. \tag{7.29}$$

We first apply Lemma 1 with $X = -Z_2^{(-1)}$, $Y = Z_2^{(n)}$, and $h(\lambda) = e^{\lambda t}$ to see that Γ_2 contains the spectrum of $Z_2^{(n)}$ in its interior for $\epsilon < \omega$ and

$$\| e^{tZ_2^{(n)}} - e^{tZ_2^{(-1)}} \| \leqslant \frac{| \partial\Gamma_2 | \nu_2 \gamma_2 \delta^{(-1)}}{2\pi(1 - \delta^{(-1)})} = O(\nu_2 \epsilon), \qquad \epsilon \to 0, \tag{7.30}$$

where

$$\nu_2 = \max_{\lambda \in \partial\Gamma_2} | e^{-\lambda t/\epsilon} | . \tag{7.31}$$

Next we apply Lemma 1 with $X = Z_2^{(n)}$, $Y = Z_2$, and $h(\lambda) = e^{\lambda t}$

$$\| e^{tZ_2} - e^{tZ_2^{(n)}} \| \leqslant \frac{| \partial\Gamma_2 | \nu_2 \gamma_2 \delta_2^{(n)}}{2\pi(1 - \delta_2^{(n)})} = O(\nu_2 \epsilon^{n+2}), \qquad \epsilon \to 0. \tag{7.32}$$

If $\partial\Gamma$ lies in the right half plane

$$R_\beta = \{\lambda \mid \operatorname{Re} \lambda \geqslant \beta\}, \qquad \beta > 0. \tag{7.33}$$

then

$$\nu_2 \leqslant e^{-\beta t}. \tag{7.34}$$

As for the outer root Z_1, one can apply the same type of argument to show that

$$\| e^{tZ_1} - e^{tZ_1^{(n)}} \| \leqslant \frac{| \partial\Gamma_1 | \nu_1 \gamma_1 \delta_1^{(n)}}{2\pi(1 - \delta_2^{(n)})} = O(\nu_1 \epsilon^{n+1}), \qquad \epsilon \to 0, \tag{7.35}$$

where Γ_1 is a Cauchy domain containing the spectrum of C;

$$\gamma_1^{(n)} = \max_{-\lambda\in\partial\Gamma_1} \|(\lambda I - Z_1^{(n)})^{-1}\|; \tag{7.36}$$

$$\delta_1^{(n)} = \| Z_1 - Z_1^{(n)} \| \gamma_1^{(n)} \leqslant (1-d)^n \epsilon \| \hat{A}_1 \| \| \hat{C}_1 \| \gamma_1/d = O(\epsilon^{n+1}), \qquad \epsilon \to 0; \tag{7.37}$$

$$\nu_1 = \max_{\lambda\in\partial\Gamma_1} e^{-\lambda t}. \tag{7.38}$$

In particular, if

$$a^* = \max_{\lambda\in\partial\Gamma_1} |\lambda|, \tag{7.39}$$

then

$$\| e^{tZ_1} - e^{tZ_1^{(n)}} \| \leqslant \frac{|\partial\Gamma_1| \gamma_1\delta_1^{(n)} e^{a^*t}}{2\pi(1-\delta_2^{(n)})} = O(\epsilon^{n+1}e^{a^*t}), \qquad \epsilon\to 0, \quad t>0. \tag{7.40}$$

Since a^* can be chosen arbitrarily close to $\| \hat{C} \|$ the inequality (7.40) is essentially (cf. (7.20))

$$\| e^{tZ_1} - e^{tZ_1^{(n)}} \| = O(\epsilon^{n+1}e^{at}), \qquad \epsilon\to 0, \quad t>0, \tag{7.41}$$

which follows from (6.5) where here $s = \| \hat{A}_1 \| \| \hat{C} \| \epsilon$.

These estimates, (7.40) an (7.41), are effective for t varying in a bounded interval. For the case $t\to\infty$ we assume that Γ_1 may be chosen in the half plane (7.33), and, therefore,

$$\nu_1 \leqslant e^{-\beta t}, \qquad t>0. \tag{7.42}$$

LEMMA 4. *Let the operators $A_1^{-1}B$ and $B^{-1}C$ have spectrum in a right half plane* (7.33) *then for ϵ sufficiently small every solution $x(t)$ of*

$$\epsilon A_1 \frac{d^2}{dt^2} x(t) + B\frac{d}{dt} x(t) + Cx(t) = 0, \qquad t>0, \tag{7.43}$$

$$\| x(t)\| = O(e^{-\beta t}) \qquad as \qquad t\to 0, \tag{7.44}$$

uniformly in ϵ.

Proof. Beginning with the representation (7.18), we apply the estimates (7.30), (7.32), (7.34), (7.35), and (7.42), and we obtain

$$x(t) = e^{-t\hat{C}}c_1 + e^{-tG}c_2 + O(\epsilon e^{-\beta t}), \qquad \epsilon\to 0, \quad t\to\infty, \tag{7.45}$$

where

$$G = \hat{A}_1^{-1}/\epsilon \tag{7.46}$$

has spectrum in R_β for $0 < \epsilon < 1$. The conclusion now follows from the Taylor–Dunford formula (cf. 3.4):

$$\begin{aligned} \| e^{-tG} \| &= \left\| \frac{1}{2\pi i} \int_{\partial\Gamma} e^{-\lambda t}(\lambda I - G)^{-1}\, d\lambda \right\| \\ &\leqslant \frac{|\partial\Gamma|}{2\pi} e^{-\beta t} \max_{\lambda\in\Gamma} \|(\lambda I - G)^{-1}\| \\ &= O(e^{-\beta t}), \qquad t \to \infty, \end{aligned} \tag{7.47}$$

and a similar estimate for $e^{-t\hat{C}}$.

Equation (7.42) may be regarded as governing the motion of a small displacement of a fluid in equilibrium where ϵA_1, B, C are the respective inertial, dissipative, and gravitational forces. When these operate in a Hilbert space and are positive then the equilibrium is known to be stable. Lemma 4 may be thought of as an extension of this result [1].

We now consider the inhomogeneous case

$$\epsilon A_1 \frac{d^2}{dt^2} x(t) + B \frac{d}{dt} x(t) + Cx(t) = y(t), \qquad t > 0, \tag{7.48}$$

where $y(t)$ is continuous. We are concerned with the asymptotic behavior of the inhomogeneous contribution $x_p(t)$. For this we may use formula (2.8), where $A = \epsilon A_1$, but it is more advantageous to deal with the alternate representation (3.2); for then we need not be concerned with the spectrum of $\hat{A}_1$. Writing

$$x_p(t) = -\frac{B}{2\pi i} \int_0^t \int_{\partial\Gamma_1} e^{(t-\tau)\lambda}(\lambda I + \hat{C} + \epsilon\lambda^2 \hat{A}_1)^{-1} y(\tau)\, d\lambda\, d\tau \tag{7.49}$$

and applying Lemma 2 with Γ_1 containing the spectrum of $-\hat{C}$ we obtain the estimate,

$$x_p(t) = -\frac{B}{2\pi i} \int_0^t \left[\int_{\partial\Gamma_1} e^{(t-\tau)\lambda}(\lambda I + \hat{C})^{-1} y(\tau)\, d\lambda + O(\epsilon\nu_1) \right] d\tau. \tag{7.50}$$

If Γ_1 may be chosen so that (7.42) holds then, after integrating the order term with respect to τ, we obtain

$$\begin{aligned} x_p(t) &= -\frac{B}{2\pi i} \int_0^t \int_{\partial\Gamma_1} e^{(t-\tau)\lambda}(\lambda I + \hat{C})^{-1} y(\tau)\, d\lambda\, d\tau + O(\epsilon), \qquad \epsilon \to 0 \\ &= -B \int_0^t e^{(\tau-t)\hat{C}} y(\tau)\, d\tau + O(\epsilon), \qquad \epsilon \to 0, \end{aligned} \tag{7.51}$$

where the order term is uniform in t. If for some real α

$$y(t) = O(e^{\alpha t}) \qquad \text{as} \qquad t \to \infty, \tag{7.52}$$

then, in view of (7.42), the contour integral term is $O(e^{\alpha t})$ as $t \to \infty$.

We summarize these result in the following.

THEOREM 1. *Let A_1, B, C, A_1^{-1}, $B^{-1} \in \mathscr{R}$. Then every solution of* (7.48) *may be represented in the form:*

$$x(t) = e^{tZ_1}c_1 + e^{tZ_2}c_2 + x_p(t), \tag{7.53}$$

where c_1, c_2 are constant vectors and Z_1, Z_2 are the respective outer, inner roots of

$$\epsilon A_1 Z^2 + BZ + C = 0, \tag{7.54}$$

and $x_p(t)$ is a particular solution given by (2.8). *Z_1 and Z_2 are representative in series in ϵ as given by* (7.6) *and* (7.11). *Each of the three terms on the right of* (7.53) *has asymptotic approximations as $\epsilon \to 0$ in terms of approximate roots obtained by linear iteration as given by the estimates* (7.30), (7.31), (7.32), (7.34), (7.35), (7.38), (7.40), (7.41), (7.42), *and* (7.51). *If the spectrum of $B^{-1}C$ lies in a right half plane R_β* (cf. (7.33)), *and* (7.52) *holds, then the last term on the right of* (7.53) *is $O(\epsilon^{\alpha t})$ as $\epsilon \to 0$, $t \to \infty$ while the first term is $O(e^{-\beta t})$ as $t \to \infty$ uniformly as $\epsilon \to 0$. If in addition, the spectrum of $A_1^{-1}B$ lies in the right half plane R_β then the middle term is $O(e^{-\beta t})$ as $t \to \infty$ uniformly as $\epsilon \to 0$ and the solutions of* (7.53) *$x(t)$ satisfies*

$$\lim_{t \to \infty} \sup t^{-1} \log \| x(t) \| = \alpha - \beta$$

uniformly as $\epsilon \to 0$.

8. DISCUSSION

In Sections 6 and 7 we required that A have an inverse and throughout the paper we have confined ourselves to bounded operators. In the future, we expect to remove both these restrictions when considering examples where the coefficients are partial differential operators with respect to independent variables other than t. The results in [4], on which Lemma 3 is based, apply to this more general type of situation.

Our objective in this paper is to illustrate the applicability of operator equations to the theory of abstract ordinary differential equations and its many applications. We have restricted ourselves to the case of constant coefficients only because of its simplicity. But this is not necessary. Consider, for example, the singular equation,

$$t^2 A(t) \frac{d^2}{dt^2} x(t) + tB(t) \frac{d}{dt} x(t) + C(t) \frac{d}{dt} x(t) = 0, \tag{8.1}$$

where $A(t)$, $B(t)$, $C(t)$ are power series in t with coefficients in $\mathscr{R}$. Under certain conditions [6] the solutions are expressed in a way analogous to that used when the coefficients are scalar valued functions. That is, we look for solutions of the form,

$$x(t) = (t)^Z p(t), \tag{8.2}$$

where $p(t)$ is a power series in t with coefficients in $\mathscr{B}$ and Z is an operator in $\mathscr{R}$. This approach will lead to solving the operator equation

$$A(0)\, Z(Z - I) + B(0)\, Z + C(0) = 0 \tag{8.3}$$

for Z. Once we have found Z we may then proceed formally as in the scalar case. If we approximate Z, say by the application of Lemma 3, then we obtain approximate solutions.

References

1. E. M. Barston, An energy principle of dissipative fluids, *J. Fluid Mech.* **42** (1970), 97–109.
2. J. D. Cole, "Perturbation Methods in Applied Mathematics," Blaisdell, Waltham, MA, 1968.
3. S. D. Conte, The numerical solution of linear boundary value problems, *SIAM Rev.* **8** (1966), 309–321.
4. J. Eisenfeld, Operator equations and nonlinear eigenparameter problems, *J. Functional Analysis* **12** (1973), 475–490.
5. J. Eisenfeld, On symmetrization and roots of quadratic eigenvalue problems, *J. Functional Analysis* **9** (1972), 410–422.
6. E. Hille, "Lectures on Ordinary Differential Equations," Addison Wesley, Reading, MA, 1969.
7. E. Hille and R. S. Phillips, "Functional Analysis and Semi-Groups," American Mathematics Society, Providence, RI, 1957.
8. M. G. Krein and H. Langer, On some mathematical principles of linear theory of damped vibrations of continua, in *Appl. of the Thy. of Functions in Cont. Mech.*, Proc. Int. Symposium (Sept. 17–23, 1963), Vol. 2, Nauka, Moscow (1965), pp. 283–322. (In Russian).
9. H. Langer, Uber stark gedampfte Scharen im Hilbertraum, *J. Math. Mech.* **17** (1967–68), 685–705.
10. L. Lapidus and J. H. Seinfeld, "Numerical Solutions of Ordinary Differential Equations," Academic Press, New York, 1971.